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Dedicated to Marius Sophus Lie in the Centennial of His Death

ISOTOPIC, GENOTOPIC AND HYPERSTRUCTURAL LIFTINGS OF LIE'S THEORY AND THEIR ISODUALS

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Abstract

After reviewing the basic role of Lie's theory for the mathematics and physics of this century, we identify its limitations for the treatment of systems beyond the local-differential, Hamiltonian and canonical-unitary conditions of the original conception. We therefore outline three sequential generalized mathematics introduced by the author under the name of *iso-*, *geno-* and *hyper-mathematics* which are based on generalized, Hermitean, non-Hermitean and multi-valued units, respectively. The resulting *iso-*, *geno-* and *hyper-Lie theories*, for which the new mathematics were submitted, have been extensively used for the description of nonlocal-integral systems with action-at-a-distance Hamiltonian and short-range-contact non-Hamiltonian interactions in reversible, irreversible and multi-valued conditions, respectively. We then point out that conventional, *iso-*, *geno-* and *hyper-Lie theories* are unable to provide a consistent *classical* representation of *antimatter* yielding the correct charge conjugate states at the operator counterpart. We therefore outline yet novel mathematics proposed by the author under the names of *isodual conventional*, *iso-*, *geno-* and *hyper-mathematics*, which constitute anti-isomorphic images of the original mathematics characterized by *negative-definite units and norms*. The emerging isodual generalizations of Lie's theory have permitted a novel consistent characterization of antimatter at all levels of study, from Newton to second quantization. The main message emerging after three decades of investigations is that the sole generalized theories as invariant as the original formulation, the sole usable in physics, are those preserving the original abstract Lie axioms, and merely realizing them in broader forms.

1. Majestic Consistency of Lie's Theory.

Since I was first exposed to the theory of *Marius Sophus Lie* [1] during my graduate studies in physics at the University of Torino, Italy, in the 1960's, I understood that Lie's theory has a fundamental character for the virtual entire contemporary mathematics and physics.

I therefore dedicated my research life to identify the *limitations* of Lie's theory and construct possible *generalizations* for physical conditions broader than those of the original conception. In this paper I outline the most salient aspects of this scientific journey (as representative references, see my original papers in the field [3], mathematical studies [4], physical studies [5], monographs [6], applications and experimental verifications [7-10]).

Let $F = F(a, +, \times)$ be a field of conventional numbers a (real, complex or quaternionic numbers) with conventional sum $+$, (associative) product \times) additive unit 0 and multiplicative unit I. When formulated on a Hilbert space \mathcal{H} over F , the physically most important formulation of Lie's theory is that via connected transformations of an operator A on \mathcal{H} over F in the following finite and infinitesimal forms and interconnecting conjugation

$$A(w) = U \times A(0) \times U^\dagger = e^{iX \times w} \times A(0) \times e^{-iw \times X}, \quad (1.1a)$$

$$idA/dw = A \times X - X \times A = [A, X]_{operator}, \quad (1.1b)$$

$$e^{iX \times w} = [e^{-iw \times X}]^\dagger, X = X^\dagger, w \in F, \quad (1.1c)$$

with classical counterpart in terms of vector-fields on the cotangent bundle (phase space) with local chart (r^k, p_k) , $k = 1, 2, 3$, over F

$$A(w) = U \times A(0) \times U^t = e^{-w \times (\partial X / \partial r^k) \times (\partial / \partial p_k)} \times A(0) \times e^{w(\partial / \partial r^k) \times (\partial X / \partial p_k)}, \quad (1.2a)$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k} = [A, X]_{classical}, \quad (1.2b)$$

and unique interconnecting map given by the conventional or symplectic quantization.

As it is well known, Lie's theory is at the foundation of the mathematics of this century, including topology, vector and metric spaces, functional analysis, differential equations, algebras and groups, geometries, etc.

As it is also well known, Lie's theory is at the foundation of all physical theories of this century, including classical and quantum mechanics, particle physics, nuclear physics, superconductivity, chemistry, astrophysics, etc. In fact, whenever the parameter w represents time t , Eqs. (1.1) are the celebrated Heisenberg equations of motion in finite and infinitesimal form, while Eqs (1.2) are the classical Hamilton equations, also in their finite and infinitesimal forms. Characterization via Lie's theory of all classical and operator branches of physics then follows.

A reason for the majestic consistency of Lie's theory most important for physical applications is that of being *form invariant* under the transformations of its own class. In fact, connected Lie groups (1.1a) constitute *unitary transforms* on \mathcal{H} over F ,

$$U \times U^\dagger = U^\dagger \times U = I, \quad (1.3)$$

under which we have the following invariance laws for units, products and eigenvalue equations

$$I \rightarrow U \times I \times U^\dagger = I' = I, \quad (1.4a)$$

$$A \times B \rightarrow U \times (A \times B) \times U^\dagger = (U \times A \times U^\dagger) \times (U \times B \times U^\dagger) = A' \times B', \quad (1.4b)$$

$$H \times |\psi\rangle = E \times |\psi\rangle \rightarrow U \times H \times |\psi\rangle = (U \times H \times U^\dagger) \times (U \times |\psi\rangle) = H' \times |\psi'\rangle =$$

$$U \times E \times |\psi\rangle = E' \times |\psi'\rangle, E' = E. \quad (1.4c)$$

with corresponding invariances at the classical level here omitted for brevity.

It then follows that *Lie's theory possesses numerically invariant units, products and eigenvalues*, thus verifying the necessary condition for physically consistent applications.

2. Initial Proposals of Generalized Theories.

Despite the above majestic mathematical and physical consistency, by no means Lie's theory can represent the totality of systems existing in the universe. In fact, inspection of structures (1.1) and (1.2) reveals that, in its conventional formulation, *Lie's theory can only represent isolated-conservative-reversible systems of point-like particles with only potential-Hamiltonian internal interactions*. In fact, the point-like structure is demanded by the local-differential character of the underlying topology; the

isolated-conservative character of the systems is established by the fact that the brackets $[A, B]$ of the time evolution are totally antisymmetric, thus implying conservation laws of total quantities; the sole potential character is established by the representation of systems solely via a Hamiltonian; and the reversibility is established by the fact that all known action-at-a-distance interactions are reversible in time (i.e., their time reversal image is as physical as the original one, as it is the case for the orbit of a planet). All admissible interactions are represented via time-independent potentials V in the Hamiltonian $H = p^2/2m + V$, resulting in manifestly reversible systems.

I therefore initiated a long term research program aiming at generalizations (I called *liftings*) of Lie's theory suitable for the representation of broader systems.

The first lifting I proposed as part of my Ph.D. thesis [3a,3b] back in 1967 is that for the representation of *open-nonconservative systems*, that is, systems whose total energy H is not conserved in time, $idH/dt \neq 0$, because of interactions with the rest of the universe. This called for the formulation of the theory in such a way that its brackets are not totally antisymmetric. In this way I proposed, apparently for the first time in 1967, the broader (p-q)-parametric deformations (known in more recent times as the q-deformations),

$$A(w) = U \times A(0) \times U^\dagger = e^{iw \times p \times X} \times A(0) \times e^{-iw \times q \times X}, X = X^\dagger, \quad (2.1a)$$

$$idA/dw = p \times A \times X - q \times X \times A = (A, X)_{operator}, \quad (2.1b)$$

where p , q and $p + / - q$ are non-null parameters, with classical counterpart [3c]

$$A(w) = U \times A(0) \times U^t = e^{-w \times q \times (\partial X / \partial r^k) \times (\partial / \partial p_k)} \times A(0) \times e^{w \times p \times (\partial / \partial r^k) \times (\partial X / \partial p_k)}, \quad (2.2a)$$

$$\frac{dA}{dw} = p \times \frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - q \times \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k} = (A, X)_{classical}. \quad (2.2b)$$

Prior to releasing papers [3a] for publication, I spent about one year in European mathematical libraries to identify the algebras characterized by brackets (A, B) which resulted to be *Lie-admissible* according to Albert [2]

(a generally nonassociative algebra with product (A, B) is said to be Lie-admissible when the attached algebra with antisymmetric product $[A, B] = (A, B) - (B, A)$ is Lie). At the time of proposal [3a] only *three* papers had appeared in Lie-admissible algebras and only in the mathematical literature (see Ref. [3a]).

The (p, q) -parametric deformations (2.1), (2.2) did indeed achieve the desired objective. In fact, the total energy and other physical quantities *are not* conserved by assumption, because $idH/dt = (p - q) \times H \times H \neq 0$.

In 1968 I emigrated with my family to the U. S. A, where I soon discovered that Lie-admissible theories were excessively ahead of their time because unknown in *mathematical*, let alone physical circles. Therefore, for a number of years I had to dedicated myself to more mundane research along the preferred lines of the time.

When I passed to Harvard University in 1978 I resumed research on Lie-admissibility and proposed the most general possible (P, Q) -operator Lie-admissible theory according to the operator structures [3d]

$$A(w) = U \times A(0) \times U^\dagger = e^{iw \times X \times Q} \times A(0) \times e^{-iw \times P \times X}, X = X^\dagger, P = Q^\dagger, \quad (2.3a)$$

$$idA/dw = A \times P \times X - X \times Q \times A = (A, X)_{operator}, \quad (2.3b)$$

where P, Q , and $P + / - Q$ are non-singular matrices (or operators) such that P - Q characterizes Lie brackets, with classical counterpart [3d]

$$A(w) = U \times A(0) \times U^t = e^{-w \times (\partial X / \partial r^i) \times Q_j^i \times (\partial / \partial p_k)} \times A(0) \times e^{w \times (\partial / \partial r^i) \times P_j^i \times (\partial X / \partial p_j)}, \quad (2.4a)$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^i} \times P_j^i \times \frac{\partial X}{\partial p_j} - \frac{\partial X}{\partial r^i} \times Q_j^i \times \frac{\partial A}{\partial p_j} = (A, X)_{classical}. \quad (2.4b)$$

A primary motivation for generalizations (2.3) and (2.4) over (2.1) and (2.2) is that the latter constitute nonunitary-noncanonical transforms. The application of a nonunitary transform to Eqs. (2.1) then yields precisely Eqs. (2.3) with $P = p \times (U \times U^\dagger)^{-1}$ and $Q = q \times (U \times U^\dagger)^{-1}$, as we shall see better below. The application of any additional nonunitary transform then preserves the Lie-admissible structure. A similar case occurs for the classical counterpart.

Additional studies established that structures (2.3) constitute the most general possible transformations admitting an algebra in the infinitesimal form. In particular, the product (A, B) results to be jointly *Lie- and Jordan admissible*, although the attached Lie and Jordan algebras are more general than the conventional forms.

The latter generalized character permitted me to propose a particularization of the above Lie-admissible theory I called *Lie-isotopic* [3d,3r] , in which the brackets did verify the Lie axioms, but are more general than the conventional versions, with operator formulation

$$A(w) = U \times A(0) \times U^\dagger = e^{iX \times T \times w} \times A(0) \times e^{-iw \times T \times X}, \hat{T} = \text{hat}T^\dagger, \quad (2.5a)$$

$$idA/dw = A \times T \times X - X \times T \times A = [A, X]_{operator}, \quad (2.5b)$$

and classical counterpart [3d,3r]]

$$A(w) = e^{-w \times (\partial X / \partial r^i) \times T_j^i \times (\partial / \partial p_j)} > A(0) < e^{-w(\partial / \partial p_j) \times T_j^i \times (\partial X / \partial r^i)}, \quad (2.6a)$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^i} \times T_j^i \times \frac{\partial X}{\partial p_j} - \frac{\partial X}{\partial r^i} \times T_j^i \times \frac{\partial A}{\partial p_j} = [A, X]_{classical}. \quad (2.6b)$$

As one can see, the latter theories too are nonunitary-noncanonical, and the application of additional nonunitary-noncanonical transforms preserves the Lie-isotopic character. This establishes that transformations (2.5), (2.6) are the most general possible ones admitting a Lie algebra in the brackets of their infinitesimal versions.

3. Inconsistencies of Initial Generalizations.

Following the proposals of theories (2.1)-(2.6), I discovered that, even though *mathematically* intriguing and significant, the above Lie-isotopic and Lie-admissible theories had *no physical applications*. This is due to the fact that all the broader theories considered have a *nonunitary structure* at the operator level with a *noncanonical structure* at the classical counterpart.

In the transition from unitary to nonunitary theories, invariances (1.4) are turned into the following noninvariances,

$$U \times U^\dagger = U^\dagger \times U \neq I, \quad (3.1a)$$

$$I \rightarrow U \times I \times U^\dagger = I' \neq I, \quad (3.1b)$$

$$A \times B \rightarrow U \times (A \times B) \times U^\dagger = (U \times A \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times B \times U^\dagger) = A' \times T \times B', T = (U \times U^\dagger)^{-1}, \quad (3.1c)$$

$$H \times |\psi\rangle = E \times |\psi\rangle \rightarrow U \times H \times |\psi\rangle = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi\rangle) = H' \times T \times |\psi'\rangle = U \times E \times |\psi\rangle = E' \times |\psi'\rangle, E' \neq E, \quad (3.1d)$$

It then follows that all theories with a nonunitary structure have the following physical inconsistencies studied in detail in Refs. [12]: 1) nonunitary theories do not have invariant units of time, space, energy, etc., thus lacking any physically meaningful applications to measurements (for which the invariance of the basic units is a necessary pre-requisite); 2) nonunitary theories do not preserve in time the original Hermiticity of operators, thus having no physically acceptable observables; 3) nonunitary theories do not have invariant conventional and special functions and transforms, thus lacking unique and invariant numerical predictions; nonunitary theories violate probability and causality laws; nonunitary theories are incompatible with Galilei's and Einstein's relativities; and suffer from other serious shortcomings. Similar inconsistencies exist at the classical level.

Corresponding mathematical inconsistencies also occur [12f,12g]. In fact, nonunitary theories are generally formulated on a conventional metric or Hilbert space defined over a given field which, in turn, is based on a given unit I. But the fundamental unit is not left invariant by nonunitary transforms by conception. It then follows that the entire mathematical structure of nonunitary theories becomes inapplicable for any value of the parameters different than the initial values.

It should be noted that the above catastrophic inconsistencies also hold for any other theory departing from Lie's theory, yet formulated via conventional mathematics, such as deformations, Kac-Moody algebras, superalgebras, etc. [12].

After systematic studies I realized that the *only* possibility to reach invariant formulations of generalized Lie theories was that of *constructing new mathematics* specifically conceived for the task at hand.

Since no other mathematics was available for the representation of the broader theories here considered, as a physics I had to initiate long and

laborious mathematical studies in constructing the new mathematics, as a pre-requisite for conducting physical research.

Predictably, the task resulted to be more difficult than I suspected. In fact, after having lifted all the essential aspects of conventional mathematics (such as numbers and fields, vector and metric spaces, algebras and groups, geometries, etc.) [3s] into the needed broader form, I continued to miss the crucial invariance.

Insidiously, the problem resulted to exist where I was expect it the least, in the *ordinary differential calculus*. It was only in memoir [3i] of 1966 that I finally achieved invariance following suitable liftings of the ordinary differential calculus. The reader is therefore warned that *all papers on Lie-isotopic and Lie-admissible theories prior to memoir [3i] have no consistent physical applications because they lack invariance*.

The invariant liftings of Lie's theory which resulted from these efforts can be summarized as follows.

4. Lie-Santilli Isotheory.

The main idea [3d] is the lifting the conventional, trivial, n -dimensional unit $I = \text{diag. } (1, 1, \dots, 1)$ of Lie's theory into a real-values, nowhere singular and positive-definite $n \times n$ -dimensional matrix \hat{I} , called *isounit* (where the prefix "iso-" means "axiom-preserving"), with an unrestricted functional dependence on time t , coordinates $r = (r^k)$, momenta $p = (p_k)$, $k = 1, 2, 3$, wavefunctions ψ , and any other needed variable,

$$I = \text{diag.}(1, 1, \dots, 1) \rightarrow \hat{I}(t, r, p, \psi, \dots) = 1/\hat{T} \neq I. \quad (4.1)$$

The applicable mathematics, called *isomathematics*, is the lifting of the *totality* of conventional mathematics (with a well defined unit), without any exception known to me, into a new form admitting \hat{I} , rather than I , as the correct left and right unit. This calls for:

1) The lifting of the associative product $A \times B$ among g generic quantities A, B (such as numbers, vector-fields, operators, etc.) into the form, called *isoassociative product*, for which \hat{I} is indeed the left and right unit,

$$A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B, \hat{I} \hat{\times} A = A \times \hat{\times} \hat{I} = A; \quad (4.2)$$

2) The lifting of fields $F = F(a, +, \times)$ into the *isofields* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ of *isonumbers* $\hat{a} = a \times \hat{I}$ (isoreal, isocomplex and iso-octonionic numbers)

with *isosum* $\hat{a} \hat{+} \hat{b} = (a + b) \times \hat{I}$, *isoproduct* $\hat{a} \hat{\times} \hat{b} = (a \times b) \times \hat{I}$, *isoquotient* $\hat{a} / \hat{b} = (\hat{a} / \hat{b}) \times \hat{I}$, etc. (see [3h] for details) ;

3) The lifting of functions $f(r)$ on F into *isofunctions* $\hat{f}(\hat{r})$ on \hat{F} , such as the *isoexponentiation* $\hat{e}^{\hat{A}} = \hat{I} + \hat{A}/1! + \hat{A} \hat{\times} \hat{A}/2! + \dots = (e^{\hat{A} \times \hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T} \times \hat{A}})$, and related lifting of transforms into *isotransforms* (see [3i,3s] for details);

4) The lifting of the ordinary differential calculus into the *isodifferential calculus*, with basic rules $\hat{d}\hat{r}^k = \hat{I}_i^k \times d\hat{r}^i$, $\hat{d}\hat{p}_k = \hat{T}_k^i \times d\hat{p}_i$ (because r^k and p_k are defined on isospaces with isometrics inverse of each other), *isoderivatives* $\hat{\partial}/\hat{\partial}\hat{r}^i = \hat{T}_i^j \times \partial/\partial\hat{r}^j$, $\hat{\partial}/\hat{\partial}\hat{p}_k = \hat{I}_k^i \times \partial/\partial\hat{p}_i$, $\hat{\partial}\hat{r}^i/\hat{\partial}\hat{r}^j = \hat{\delta}_j^i = \delta_j^i \times \hat{I}$, etc.(see [3i] for details);

5) The lifting of conventional vector, metric and Hilbert spaces into their isotopic images, e.g., the lifting of the Euclidean space $E(r, \delta, R)$ with local coordinates $r = (r^k)$ and metric $\delta = \text{Diag.}(1, 1, 1)$ into the *isoeuclidean spaces* $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ with isocoordinates $\hat{r} = r \times \hat{I}$ and isometric $\hat{\delta} = \hat{T} \times \delta$ over the isoreals \hat{R} , or the lifting of the Hilbert space \mathcal{H} with inner product $\langle | \times | \rangle \times I$ over the complex field C into the *isohilbert space* $\hat{\mathcal{H}}$ with *isoinner product* $\langle | \hat{\times} | \rangle \times \hat{I}$ over the isocomplex field \hat{C} ; etc. (see [3s] for details).

6) The lifting of geometries and topologies into their corresponding isotopic images (see [3n] for details);

7) The isotopic lifting of all various branches of Lie's theory, such as the liftings of: universal enveloping associative algebras (including the Poincaré-Birkhoff-Witt theorem), Lie's algebras (including Lie's first, second and third theorems); Lie's groups, transformation and representation theory, etc.

The main operator formulation of the Lie-Santilli isothory can be written

$$\hat{A}(\hat{w}) = \hat{e}^{i\hat{X} \hat{\times} \hat{w}} \hat{\times} \hat{A}(\hat{0}) \hat{\times} \hat{e}^{-i\hat{w} \hat{\times} \hat{X}} =$$

$$[e^{i\hat{w} \times \hat{X} \times \hat{T}} \times A(0) \times e^{-i\hat{w} \times \hat{T} \times \text{hat}X}] \times \hat{I}, X = X^\dagger, \hat{T} = \hat{T}^\dagger, \quad (4.3a)$$

$$i\hat{d}\hat{A}/\hat{d}\hat{w} = \hat{A} \hat{\times} \hat{X} - \hat{X} \hat{\times} \hat{A} = \hat{A} \times \hat{T} \times \hat{X} - \hat{X} \times \hat{T} \times \hat{A} = [A; X]_{\text{operator}}, \quad (4.3b)$$

$$\text{hate}i\hat{X} \times \hat{w} = (\hat{e}^{-i\hat{w} \hat{\times} \hat{X}})^{\text{dagger}}, \quad (4.3c)$$

with classical counterpart

$$\hat{A}(\hat{w}) = e^{-w \times (\hat{\partial}\hat{X}/\hat{\partial}\hat{r}^k) \hat{\times} (\hat{\partial}/\hat{\partial}\hat{p}_k)} \hat{\times} \hat{A}(\hat{0}) \hat{\times} e^{w(\hat{\partial}/\hat{\partial}\hat{r}^k) \hat{\times} (\hat{\partial}\hat{X}/\hat{\partial}\hat{p}_k)}, \quad (4.4a)$$

$$\begin{aligned} \frac{d\hat{A}}{d\hat{w}} &= \frac{\partial\hat{A}}{\partial\hat{r}^k} \hat{\times} \frac{\partial X}{\partial\hat{p}_k} - \frac{\partial\hat{X}}{\partial\hat{r}^k} \hat{\times} \frac{\partial\hat{A}}{\partial\hat{p}_k} = \\ &= \left[\frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k} \right] \times \hat{I} = [A, X]_{\text{classical}}, \end{aligned} \quad (4.4b)$$

and unique interconnecting map called *isosymplectic quantization* [3s].

A most salient feature of the Lie-Santilli isothory is that it is *form invariant under all possible nonunitary transforms*, thus achieving the fundamental physical objective indicated earlier. In fact, an arbitrary nonunitary transform on \mathcal{H} over F can always be uniquely written as the *isounitary transform* on $\hat{\mathcal{H}}$ over \hat{F} ,

$$V \times V^\dagger = \hat{I} \neq I, V = \hat{V} \times \hat{T}^{1/2}, V \times V^\dagger = \hat{V} \hat{\times} \hat{V}^\dagger = \hat{V}^\dagger \hat{\times} \hat{V} = \hat{I}, \quad (4.5)$$

under which we have the isoinvariance laws

$$\hat{I} \rightarrow \hat{V} \times \hat{I} \times \hat{V}^\dagger = \hat{I}' = \hat{I}, \quad (4.6a)$$

$$\hat{A} \hat{\times} \hat{B} \rightarrow \hat{V} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{V}^\dagger = (\hat{V} \hat{\times} \hat{A} \hat{\times} \hat{V}^\dagger) \hat{\times} (\hat{V} \hat{\times} \hat{B} \hat{\times} \hat{V}^\dagger) = \hat{A}' \hat{\times} \hat{B}', \quad (4.6b)$$

$$\begin{aligned} \hat{H} \hat{\times} |\hat{\psi}\rangle &= \hat{E} \hat{\times} |\hat{\psi}\rangle \rightarrow \hat{V} \hat{\times} \hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{V} \hat{\times} \hat{H} \hat{\times} \hat{V}^\dagger \hat{\times} \hat{V} \hat{\times} |\hat{\psi}\rangle = \hat{H}' \hat{\times} |\hat{\psi}'\rangle = \\ &= \hat{V}' \hat{\times} \hat{E} \hat{\times} |\hat{\psi}'\rangle = \hat{E}' \hat{\times} |\hat{\psi}'\rangle, \hat{E}' = \hat{E}, \end{aligned} \quad (4.6c)$$

with corresponding isoinvariances for the classical counterpart.

As one can see, isomathematics achieves the *invariance of the numerical values of the isounit, isoproduct and isoeigenvalues*, thus regaining the necessary conditions for physical applications.

It is easy to prove that *isohermiticity coincides with the conventional Hermiticity*. As a result, all conventional observables of unitary theories remain observables under isotopies. The preservation of Hermiticity-observability in time is then ensured by the above isoinvariances. Detailed studies conducted in Ref. [3l] established the resolution of all inconsistencies of nonunitary theories.

By comparing Eqs. (1.1)-(1.2) and (4.3)-(4.4) it is evident that the Lie theory and the Lie-Santilli isothory coincide at the abstract level by conception and construction [3d,3i]. In fact, the latter can be characterized

by "putting a hat" to the totality of quantities and operations of Lie's theory with no exception known to me (otherwise the invariance is lost).

Despite this mathematical similarity, the physical implications of the Lie-Santilli isothory are far reaching. By recalling that Lie's theory is at the foundation of all of physics, Eqs. (4.3) and (4.4) have permitted a structural generalization of the fundamental dynamical equations of classical and quantum mechanics, superconductivity and chemistry into new disciplines called *isomechanics* [3] *isosuperconductivity* [7] and *isochemistry* [8]. These new disciplines essentially preserve the physical content of the old theories, including the preservation identically of the total conserved quantities, but add internal nonhamiltonian effects represented by the isounit that are outside any hope of representation via Lie's theory.

In turn, these novel effects have permitted momentous advances in various scientific fields, such as the first axiomatically consistent unification of electroweak and gravitational interactions [3k,3q].

An illustrative classical application of the Lie-Santilli isothory is the representation of the structure of Jupiter when considered isolated from the rest of the Solar system, with action-at-a-distance gravitational and other interactions represented with the potential V in the Hamiltonian H and additional, internal contact-non-Hamiltonian interactions represented via the isounit \hat{I} .

An illustrative operator application is given by novel structure models of the strongly interacting particles (called *hadrons*) for which the theory was constructed [3j]. In turn, the latter application has far reaching implications, including the prediction of novel, clean *subnuclear* energies.

5. Lie-Santilli Genothory.

The main insufficiency of the Lie-Santilli isothory is that it preserves the totally antisymmetric character of the classical and operators Lie brackets, thus being unsuited for a representation of open-nonconservative systems. In particular, despite the broadening of unitary-canonical theories into nonunitary-noncanonical extensions, the fundamental problem of the *origin of the irreversibility* of our macroscopic reality does not admit quantitative treatment via the Lie-Santilli isothory because the latter theory is also structurally reversible (that is, the theory coincides with its time

reversal image for reversible Hamiltonians and isounits).

The resolution of this insufficiency required the broadening of the Lie-Santilli isothory into a form whose brackets are neither totally antisymmetric nor totally symmetric. In turn, the achievement of an invariant formulation of the latter theory requires the construction of a new mathematics I suggested back in 1978 [3d] under the name of *genomathematics* (where the prefix "geno" now stands for "axiom-inducing").

The main idea of genomathematics is the selection of *two different generalized units* called *genounits*, the first $\hat{I}^>$ for the *ordered multiplication to the right* $A > B$, called *forward genoproduct*, and the second $^<\hat{I}$ for the *ordered multiplication to the left* $A < B$, called *backward genoproduct*, according to the general rules [3d,3i,3l]

$$\hat{I}^> = 1/\hat{S}, A > B = A \times \hat{S} \times B, \hat{I}^> > A = A > \hat{I}^> = A, \quad (5.1a)$$

$$^<\hat{I} = 1/\hat{R}, A < B = A \times \hat{R} \times B, ^<\hat{I} < A = A < ^<\hat{I} = A, \quad (5.1b)$$

$$A = A^\dagger, B = B^\dagger, \hat{R} = \hat{S}^\dagger \quad (5.1c)$$

The broader genomathematics is then given by:

1) The lifting of isofields $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ into the *forward and backward genofields* $\hat{F}^>(\hat{a}^>, \hat{+}^>, >)$ and $^<\hat{F}(^<\hat{a}, ^<\hat{+}, ^<)$ with *forward and backward genonumbers* $\hat{a}^> = a \times \hat{I}^>$ and $^<\hat{a} = ^<\hat{I} \times a$, and related operations [3h];

2) The lifting of isofunctions $\hat{f}(\hat{r})$ on \hat{F} into the *forward and backward genofunctions* $\hat{f}^>(\hat{r}^>)$ and $^<\hat{f}(^<\hat{r})$ on $\hat{F}^>$ and $^<\hat{F}$, respectively, such as $\hat{e}_{>}^{\hat{X}} = (e^{\hat{X} \times \hat{R}}) \times \hat{I}^>$ and $\hat{e}_{<}^{\hat{X}} = ^<\hat{I} \times e^{\hat{S} \times ^<\hat{X}}$, with consequential genotopies of transforms and functional analysis at large [3i,3s];

3) The lifting of the isodifferential calculus into the *forward and backward genodifferential calculus* with main forward rules $\hat{d}^>\hat{r}^{>k} = \hat{I}_i^{>k} \times d\hat{r}^{>i}, \hat{d}^>\hat{p}_k^{>} = \hat{T}_k^{>i} \times d\hat{p}_i^{>}, \hat{\partial}^>/\hat{\partial}^>\hat{r}^{>i} = \hat{S}_i^{>j} \times \partial/\partial\hat{r}^{>j}, \hat{\partial}^>/\hat{\partial}^>\hat{p}_k^{>} = \hat{S}_k^{>i} \times \partial/\partial\hat{p}_i^{>}, \hat{\partial}^>\hat{r}^{>i}/\hat{\partial}^>\hat{r}^{>j} = \hat{\delta}_j^{>i} = \delta_j^i \times \hat{I}^>$, etc., and corresponding backward rules easily obtainable via conjugation (see [3i] for details);

4) The lifting of isotopologies, isogeometries, etc. into the dual forward and backward genotopic forms; and

5) The lifting of the Lie-Santilli isothory into the genothory, including the genotopies of the various aspects, such as universal enveloping associative algebras for ordered product to the right and to the left, etc. [3i,3r,3s].

The explicit realization of the Lie-Santilli genotheory can be expressed via the following finite and infinitesimal forms with related interconnection (at a fixed value of the parameter w , thus without its ordering) [3i,3l]

$$\begin{aligned} \hat{A}(\hat{w}) &= e_{>}^{i\hat{X}>>\hat{w}} > \hat{A}(\hat{0}) < e_{<}^{-i\hat{w}<<\hat{X}} = \\ [e^{i\hat{X}\times\hat{S}\times w} \times \hat{I}^>] \times \hat{S} \times \hat{A}(\hat{0}) \times \hat{R} \times [<\hat{I} \times e_{<}^{-i\hat{w}\times\hat{R}\times\hat{X}}], \end{aligned} \quad (5.2a)$$

$$\begin{aligned} id\hat{A}/d\hat{w} &= \hat{A} < \hat{X} - \hat{X} > \hat{A} = \\ \hat{A} \times \hat{R} \times \hat{X} - \hat{X} \times \hat{S} \times \hat{A} &= (\hat{A}, \hat{X})_{operator}, \end{aligned} \quad (5.2b)$$

$$<\hat{X} = (\hat{X}^>)^\dagger, \hat{R} = \hat{S}^\dagger \quad (5.2c)$$

classical counterpart [3i]

$$\begin{aligned} \hat{A}(\hat{w}) &= \hat{e}_{>}^{-\hat{X}>>\hat{w}} > \hat{A}(\hat{0}) < \hat{e}_{<}^{\hat{w}<<\hat{X}} = \\ e^{-w \times (\hat{\partial}^>\hat{X}^>/\hat{\partial}^>\hat{r}^>k) > (\hat{\partial}^>/\hat{\partial}^>\hat{p}_k^>)} > \hat{A}(\hat{0}) < \hat{e}^{w(\hat{<}\partial/\hat{<}\hat{\partial}^<k\hat{r}) < (\hat{<}\hat{\partial}^<\hat{X}/\hat{<}\hat{\partial}_k^<\hat{p})}, \end{aligned} \quad (5.3a)$$

$$\begin{aligned} \frac{d\hat{A}}{d\hat{w}} &= < \frac{\hat{\partial}^<\hat{A}}{\hat{\partial}^<k\hat{r}} < \frac{\hat{<}\partial X}{\hat{\partial}_k^<\hat{p}} - \frac{\hat{\partial}^>\hat{X}^>}{\hat{\partial}^>\hat{r}^>k} > \frac{\hat{\partial}^>\hat{A}^>}{\hat{\partial}^>\hat{p}_k^>} = \\ <\hat{I} \times [\frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k}] - [\frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k}] \times \hat{I}^> &= (A, X)_{classical} \end{aligned} \quad (5.3b)$$

with unique interconnecting map called *genosymplectic quantization* [3s].

A most important feature of the Lie-Santilli genotheory is its *form invariance*. This can be seen by noting that a *pair* of nonunitary transforms on \mathcal{H} over \hat{C} can always be identically rewritten as the *genounitary transforms* on genohilbert spaces over genocomplex fields,

$$V \times V^\dagger \neq 1, V = < \hat{V} \times \hat{R}^{1/2}, V \times V^\dagger = < \hat{V} < < \hat{V}^\dagger = < \hat{V}^\dagger < < \hat{V} = < \hat{I}, \quad (5.4a)$$

$$W \times W^\dagger \neq 1, W = \hat{W}^> \times \hat{S}^{1/2}, W \times W^\dagger = \hat{W}^> > \hat{W}^>\dagger = \hat{W}^>\dagger > \hat{W}^> = \hat{I}^>, \quad (5.4b)$$

under which we have indeed the following forward genoinvariance laws [3j]

$$\hat{I}^> \rightarrow \hat{I}'^> = \hat{W}^> > \hat{I}^> > \hat{W}^>\dagger = \hat{I}^>, \quad (5.5a)$$

$$\hat{A} > \hat{B} \rightarrow \hat{W}^> > (\hat{A} > \hat{B}) > \hat{W}^{>\dagger} = \hat{A}' > \hat{B}', \quad (5.5b)$$

$$\begin{aligned} \hat{H}^> > |> = \hat{E}^> > |> = E \times |> \rightarrow \hat{W}^> > \hat{H}^> > |> = \hat{H}'^> > |>' = \\ \hat{W}^> > \hat{E}^> > |> = E \times |>', \end{aligned} \quad (5.5c)$$

with corresponding rules for the backward and classical counterparts.

The above rules confirm the achievement of the *invariance of the numerical values of genounits, genoproducts and genoeigenvalues*, thus permitting physically consistent applications.

By recalling again that Lie's theory is at the foundation of all of contemporary science, the Lie-Santilli genothory has permitted an additional structural generalization of classical and quantum isomechanics, isosuperconductivity and isochemistry into their genotopic coverings.

Intriguingly, the product $\hat{A} < \hat{B} - \hat{B} > \hat{A} = \hat{A} \times \hat{R} \times \hat{B} - \hat{B} \times \hat{S} \times \hat{A}$, $\hat{R} \neq \hat{S}$, is manifestly non-Lie on conventional spaces over conventional fields, yet it becomes fully antisymmetry and Lie when formulated on the bimodule of the respective envelopes to the left and to the right, $\{<\hat{\mathcal{A}}, \hat{\mathcal{A}}>\}$ (explicitly, the numerical values of $\hat{A} < \hat{B} = \hat{A} \times \hat{R} \times \hat{B}$ computed with respect to $<\hat{I} = 1/\hat{R}$ is the same as that of $\hat{A} > \hat{B} = \hat{A} \times \hat{S} \times \hat{B}$ when computed with respect to $\hat{I}^> = 1/\hat{S}$) [3i,3l].

A primary feature of the broader classical and operator genotheories is that it represents open-nonconservative systems, as desired, because now the total energy H is not conserved in our spacetime, $idH/dt = H \times (\hat{R} - \hat{S}) \times H \neq 0$. Yet, the notion of *genohermiticity* on $\hat{\mathcal{H}}^>$ over $\hat{C}^>$ coincides with conventional Hermiticity. Therefore, the Lie-admissible theory provides the only operator representation of open systems known to this author in which the *nonconserved Hamiltonian and other quantities are Hermitean, thus observable*. In other treatments of nonconservative systems the Hamiltonian is generally *nonhermitean* and, therefore, *not observable*.

More importantly, genotheories have permitted a resolution of the historical problem of the *origin of irreversibility* via its reduction to the ultimate possible layers of nature, such as particles in the core of a star. The interested reader can find the invariant genotopic formulations of: Newton's equations in Ref. [3i]; Hamilton's equations with external terms in Ref. [3i]; quantization for open-irreversible systems in Ref. [3i,3l]; operator theory of open-irreversible systems in Ref. [3l].

6. Lie-Santilli Hypertheory.

By no means genotheries are sufficient to represent the entirety of nature, e.g., because they are unable to represent *biological structures* such as a cell or a sea shell. The latter systems are indeed open-nonconservative-irreversible, yet they possess a structure dramatically more complex than that of a nonconservative Newtonian system. A study of the issue has revealed that the limitation of genotheries is due to their *single-valued character*.

As an illustration, mathematical treatments complemented with computer visualization [10] have established that the *shape* of sea shells can be well described via the conventional single-valued three-dimensional Euclidean space and geometry according to the empirical perception of our three Eustachian tubes. However, the same space and geometry are basically insufficient to represent *the growth in time* of sea shells. In fact, computer visualization shows that, under the exact imposition of the Euclidean axioms, sea shells first grow in time in a distorted way and then crack.

Illert [10] showed that a minimally consistent representation of the sea shells growth in time requires *six dimensions*. But sea shells exist in our environment and can be observed via our *three-dimensional* perception. The solution of this apparent dichotomy I proposed [10] is that via *multi-valued hypermathematics* essentially characterized by the relaxation of the single-valued nature of the genounits while preserving their nonsymmetric character (as a necessary condition to represent irreversible events), according to the rules [3i,3t]

$$\hat{I}^> = \{\hat{I}_1^>, \hat{I}_2^>, \hat{I}_3^>, \dots\} = 1/\hat{S}, \quad (6.1a)$$

$$A > B = \{A \times \hat{S}_1 \times B, A \times \hat{S}_2 \times B, A \times \hat{S}_3 \times B, \dots\}, \hat{I}^> > A = A > \hat{I}^> = A \times I, \quad (6.1b)$$

$$<\hat{I} = \{<\hat{I}_1, <\hat{I}_2, <\hat{I}_3, \dots\} = 1/\hat{R}, \quad (6.1c)$$

$$A < B = \{A \times \hat{R}_1 \times B, A \times \hat{R}_2 \times B, A \times \hat{R}_3 \times B, \dots\} <\hat{I} < A = A <<\hat{I} = I \times A, \quad (6.1d)$$

$$A = A^\dagger, B = B^\dagger, \hat{R} = \hat{S}^\dagger. \quad (6.1e)$$

All aspects of the bimodular genotheries admit a unique, and significant extension to the above hyperstructures and their explicit form is here

omitted for brevity [3i,3t]. The expression of the theory via *weak equalities and operations* was first studied by Santilli and Vougiouklis in Ref. [11].

7. Isodual theories.

Mathematicians appear to be unaware of the fact that, contrary to popular beliefs, *the totality of contemporary mathematics, including its isotopic, genotopic and hyperstructural liftings, cannot provide a consistent classical representation of antimatter*. In fact, all these mathematics admit *only one quantization channel*. As a result, the operator image of any classical treatment of antimatter via these mathematics simply cannot yield the correct charge conjugate state, but it merely yields a particle with the wrong sign of the charge.

The occurrence should not be surprising because the study of antimatter constitutes one of the biggest scientific unbalances of this century. In fact, matter is treated at all possible mathematical and physical levels, from Newton's equations and underlying topology, all the way to second quantization and quantum field theories, while antimatter is solely treated at the level of *second quantization*. However, astrophysical evidence suggests quite strongly the existence of macroscopic amounts of antimatter in the universe, to the point that even entire galaxies and quasars could eventually result to be made up entirely of antimatter.

The only possible resolution of this historical unbalance is that via the construction of a *yet new mathematics*, specifically conceived for a consistent *classical* representation of antimatter whose operator counterpart yields indeed the correct charge conjugate states.

Recall that charge conjugation is anti-homomorphic, although solely applies at the operator level. It then follows that the new mathematics for antimatter should be, more generally, anti-isomorphic and applicable at all levels of study.

After a laborious research, I proposed back in 1985 [3g] the *isodual mathematics*, namely, mathematics constructed via the *isodual map* of numbers, fields, spaces, algebras, geometries, etc..

The *isodual conventional mathematics* is characterized by the simplest conceivable anti-isomorphic map of the unit into its *negative-definite form*,

$$I > 0 \rightarrow -I = I^d < 0, \quad (7.1)$$

under which we have the transformation law of a generic, scalar, real-valued quantity

$$A(w) \rightarrow A^d(w^d) = -A(-w), \quad (7.2)$$

with reconstruction of numbers, fields, spaces, algebras, geometries, quantization, etc. in such a way to admit I^d , rather than I , as the correct left and right unit.

The isodual map characterizing the broader *isodual iso-, geno- and hyper-mathematics* is instead given by

$$\hat{I}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \dots) \rightarrow -\hat{I}^\dagger(-\hat{t}^\dagger, -\hat{r}^\dagger, -\hat{p}^\dagger, -\hat{\psi}^\dagger, \dots) = \hat{I}^d(\hat{t}^d, \hat{r}^d, \hat{p}^d, \hat{\psi}^d, \dots), \quad (7.3)$$

and consequential reconstruction of the entire formalism to admit \hat{I}^d as the correct left and right new unit.

The above map is not trivial, e.g., because it implies the reversal of the sign of *all* physical characteristics of matter (and not only of the charge). As such, isodual theories provide a novel intriguing representation of antimatter which *begins* at the primitive classical Newtonian level, as desired, and then persists at all subsequent levels, including that of second quantization, in which case isoduality becomes equivalent to charge conjugation [3m].

The most general mathematics presented in this paper is the *isoselfdual hypermathematics* [3i], namely, a hypermathematics that coincides with its isodual, and is evidently given by hypermathematics multiplied by its isodual. The latter mathematics has been used for one of the most general known cosmologies [3p] inclusive of antimatter as well as of biological structures (as any cosmology should be), in which the universe: has a multi-valued structure perceived by our Eustachian tubes as a single-valued three-dimensional structure; admits equal amounts of matter and antimatter (in its limit formulation verifying Lies conjugation (1.1c)); removes any need for the "missing mass"; reduces considerably the currently believed dimension of the measured universe; possesses all *identically null* total characteristics of time, energy, linear and angular momentum, etc.; eliminates any singularity at the time of creation.

8. Simple Construction of Generalized Theories.

Unpredictably, the need for new mathematics has been a major obstacle

for the propagation of the generalized Lie theories outlined in this paper in both mathematical and physical circles.

I would like to indicate here that *all* generalized Lie theories, *all* their underlying new mathematics and *all* their applications can be uniquely and unambiguously constructed via the following elementary means accessible to undergraduate students.

First, isotheries can be constructed via the systematic application of the following nonunitary transform

$$U \times U^\dagger = \hat{I}, (U \times U^\dagger)^{-1} = \hat{T}, \quad (8.1)$$

to the *totality* of the original formalism with no exceptions.

In fact, transform (8.1) yields the isonumbers $U \times n \times U^\dagger = n \times \hat{I}$, the isoproduct, $U \times (A \times B) \times U^\dagger = (U \times A \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times B \times U^\dagger) = A' \times \hat{T} \times B' = A' \hat{\times} B'$; the correct isofunctions, such as $U \times e^X \times U^\dagger = \hat{e}^{\hat{X}}$; and the correct expression of all other aspects, including the Lie-Santilli isothery and its underlying basic theorems.

Once the isotopic structure has been achieved in this way, its invariance is proved via the reformulation of nonunitary transforms in the isounitary form (4.5), with consequential invariance of the isothery as in Eqs. (4.6).

The construction of the Lie-Santilli genotheory is equally elementary, and requires the use, this time, of *two* nonunitary transforms

$$U \times U^\dagger \neq I, W \times W^\dagger \neq I, U \times W^\dagger = \hat{I}^>, W \times U^\dagger = \hat{I}^<, \quad (8.2)$$

to the *totality* of the original formalism, again, without any exceptions.

In fact, transforms (8.2) yields the correct form of forward and backward genonumbers, e.g., $U \times n \times W^\dagger = n \times \hat{I}^>$, the correct form of the forward and backward genoproduct, genofunctions and genotransforms, including the correct structure and representation of the Lie-Santilli genotheory. Once reached in this way, the invariance is proved by rewriting the nonunitary transforms in their genounitary version (5.4). Genoinvariant laws (5.5) then follow.

The Lie-Santilli hypertheory can be constructed and proved to be invariant via the mere relaxation of the single-valued character of the genounits. The explicit construction is here omitted for brevity [3t]).

Finally, the isodual Lie theory can be easily constructed via the systematic application of the anti-isomorphic transform

$$U \times U^\dagger = -I = I^d, \quad (8.3)$$

to the totality of the original formalism with no exceptions.

This yields isodual numbers, fields, products, functions, etc. The isodualities of isotopic, genotopic and hyperstructural theories can be similarly constructed via the anti-isomorphic images of the preceding transforms.

Note that the above methods is useful on both mathematical and physical grounds. On mathematical grounds one can start from one given structure, e.g., the representation of the conventional Poincaré symmetry and construct explicitly all infinitely possible irreps of the Poincaré-Santilli iso-, geno- and hyper-symmetries as well as their isoduals [3,4].

The methods is also useful for the ongoing efforts to unify all simple Lie groups of the same dimension in Cartan's classification (over a field of characteristic zero) into one single isogroup, whose study has been initiated by Gr. Tsagas and his group [4].

On physical grounds, the method presented in this section is also particularly valuable to generalize existing applications of Lie's theory via the appropriate selection of the nonunitary transform representing the missing characteristics or properties, e.g., the representation of a locally varying speed of light.

9. Ultimate Significance of Lie's Axioms.

A unitary Lie group has the structure of a *bi-module* in both its finite and infinitesimal forms with an action from the left $U^> = e^{iX \times w}$ and an action from the the right $<U = e^{-iw \times X}$ interconnected by Hermitean conjugation (1.1c) [3e]. Eqs. (1.1) can then be written

$$A(w) = U^> > A(0) << U = e^{iX \times w} > A(0) < e^{-iw \times W}, \quad (9.1a)$$

$$idA/dt = A < X - X > A, \quad (9.1b)$$

$$<U = (U^>)^\dagger, X = X^\dagger. \quad (9.1c)$$

In the Lie case both products $A < B$ and $A > B$ are evidently conventional associative products, $A < B = A > B = A \times B$, resulting in Lie's

bimodule. However, axiomatic structure (9.1) *does not* require that such products have necessarily to be conventionally associative, because they can also be isoassociative, thus yielding the Lie-Santilli isothory. Moreover, axioms (9.1) *do not* require that the forward and backward isoassociative products have to be necessarily the same, because they can also be different, provided that conjugation (9.1c) is met. In the latter case the axioms yield the Lie-Santilli genothory with an easy extension to the hypertheory via multi-valued realizations. Isodual theories emerge along similar lines because axioms (9.1) do not necessarily demand that the underlying unit be positive-definite.

It then follows that the axiomatic consistency and invariance of the generalized theories studied in this paper can be inferred from the original invariance of Lie's theory itself, of course, when treated with the mathematics leaving invariant the basic units. The only applicable mathematics are then the iso-, geno-, and hyper-mathematics and their isoduals.

In conclusion, by looking in retrospect some three decades of studies on the topics outlined in this paper, the emerging most important message is that *the sole invariant classical and operator theories are those preserving the abstract Lie axioms*, Eq.s (1.1) and (1.2), *and merely providing their broder realizations treated with the appropriate mathematics.*

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